arXiv:hep-th/9703167v2 25 Mar 1997

Global Anomalies and Geometric Engineering of Critical Theories in Six Dimensions

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ABSTRACT

We show the existence of global gauge anomalies in six dimensions for gauge groups $SU(2), SU(3)$ and G_2 coupled to matter, characterized by an element of Z_{12}, Z_6 and Z_3 respectively. Consideration of this anomaly rules out some of the recently proposed 6 dimensional $N = 1$ QFT's which were conjectured to possess IR fixed point at infinite coupling. We geometrically engineer essentially all the other models with one tensor multiplet using F-theory. In addition we construct 3 infinite series using F-theory geometry which do not have field theory analogs. All these models in the maximally Higgsed phase correspond to the strong coupling behaviour of $E_8 \times E_8$ heterotic string compactification on $K3$ with instanton numbers $(12 + n, 12 - n)$.

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The aim of this paper is twofold. First we discuss the existence of global gauge anomalies for six dimensional theories with gauge groups $SU(2), SU(3), G_2$ which puts restrictions on their matter content, generalizing Witten's $SU(2)$ anomaly in $d = 4$ [1]. Next we specialize this to the case of $N = 1$ theories in $d = 6$ and find how this restriction automatically arises from the geometry of F-theory compactifications on Calabi-Yau threefolds [2]. Moreover for the case of simple gauge groups we discuss how the $N = 1$ superconformal theories conjectured to exist in [3] and classified in [4] are realized in F-theory, after we take into account the above global anomaly. Moreover we construct three additional infinite series using F-theory which have no field theory analog and correspond to a generalization of the phenomenon of zero size E_8 instantons. All these theories in the maximally higgsed phase are equivalent to strong coupling singularity of heterotic strings for $E_8 \times E_8$ gauge group with instanton numbers $(12 + n, 12 - n)$, for $1 \le n \le 12$. These critical theories, in the maximally higgsed phases have already been considered in [5].

1. Global Gauge Anomalies in 6 dimensions

We consider gauge theories in 6 dimensions with gauge group G with some matter field. Our considerations will be general and in particular do not depend on having any supersymmetries. We will be interested in fermionic matter in these theories. Let us call spinors of opposite chirality $S^{\pm}$. We will assume that (S^+, S^-) spinors transform according to some representation (R^+, R^-) of G respectively. If R^+ is not the same as R^- then the determinant of fermions will involve phases and defining them in principle can lead to local or global anomalies. We will start from a situation where local anomalies cancel (via a Green-Schwarz mechanism) and are interested in knowing if there are any global gauge anomalies. Such an anomaly can arise if the space of gauge transformations is disconnected.

A well known example of this occurs in 4 dimensions for $SU(2)$ gauge groups. Since $\pi_4(SU(2)) = \mathbf{Z_2}$ taking the space to be S^4 one can consider a gauge transformation not continuously connected to identity [1] and one can show that if we have an odd number of fermion doublets the fermion determinant picks up a minus sign.

One way to prove the $SU(2)$ anomaly, which will immediately generalize to the case under consideration is to embed the $SU(2)$ group in a bigger group for which π_4 is trivial but which may have local gauge anomalies. In this way the global anomaly of $SU(2)$ will be related to the local anomaly of the higher dimensional group which is technically easier to

deal with. This is the approach followed in [6]. They consider embedding $SU(2)$ into $SU(3)$ for which $\pi_4(SU(3)) = 0$. One starts with $SU(3)$ with a single fundamantal Weyl fermion and considers the non-trivial $SU(2)$ gauge transformation g on the four dimensional space, which is taken to be S^4. In the $SU(3)$ embedding this can be extended to a pure gauge transformations on the disc D^5 whose boundary is S^4. The $SU(3)$ theory with one Weyl fermion in the fundamental representation is anomalous and the variation of the phase of the determinants can be expressed as an integral over D^5

$$Z \to Z \exp(\mathrm{i} \int_{D^5} \gamma_5)$$

where $\gamma_5(g, A)$ is a specific 5-form [6]. Moreover the fact that the $SU(2)$ theory has no *local* anomalies implies that γ_5 vanishes on $S^4 = \partial D^5$ and thus the above integral can be viewed as an integral over S^5. One considers the exact sequence

$$\pi_5(SU(3)) \to \pi_5(SU(3)/SU(2)) \to \pi_4(SU(2)) \to \pi_4(SU(3))$$

$$\mathbf{Z} \to \mathbf{Z} \to \mathbf{Z_2} \to 0,$$

which implies that the basic generator of $\pi_5(SU(3))$ gets mapped to the square of the generator of the $\pi_5(SU(3)/SU(2))$ and that the generator of our anomaly is in the image of the generator of $\pi_5(SU(3)/SU(2))$. Using the fact that γ_5 is proportional to $Tr(g^{-1}dg)^5 + d\eta$ (for some (computable) 4-form η which depends on g and the gauge connection A) such that for the basic generator of $\pi_5(SU(3))$ the integral comes out to be 2π, and the additivity property of $\int \gamma_5$ one immediately learns that the global anomaly for $SU(2)$ for a single doublet is $exp(i\pi)$.

In fact exactly the same method works in higher dimensions, which was one of the main motivations of [6] for developing it. The case of $SU(2)$ in 4 dimensions is part of the general class considered in [6] where $2n$ dimensional theories with gauge group $SU(n)$ ($\pi_{2n}(SU(n)) = \mathbf{Z}_{n!}$) were considered. The same arguments lead to the conclusion that if we have a theory with $SU(n)$ gauge theory which is free of local anomalies and which can be embedded in $SU(n+1)$ representation R, then there is a global discrete anomaly leading to the phase

$$\exp(\frac{2\pi i A_R}{n!}) \tag{1.1}$$

where A_R is defined by

$$Tr_R F^{n+1} = A_R Tr_{\mathrm{f}} F^{n+1} + \text{lower terms} \tag{1.2}$$

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where Tr_{f} refers to trace in the fundamental representation. Note that only A_R mod $n!$ is relevant for the anomaly. The lower terms in (1.2) are irrelevant as they give rise to integrals which vanish on S^{2n+1}. Let us now come to the case of 6 dimensional gauge theories where the relevant question is whether $\pi_6(G)$ is non-trivial. Indeed this is so for the groups $G = SU(2), SU(3), G_2$ where we have

$$\pi_6(SU(2)) = \mathbf{Z_{12}}$$

$$\pi_6(SU(3)) = \mathbf{Z_6}$$

$$\pi_6(G_2) = \mathbf{Z_3}$$

The fact that there could potentially exist an anomaly in these cases was anticipated in [7][8]. As noted above the case of $SU(3)$ is a special case of the situation considered in [6]. However this case was ruled out in [6] because local anomaly cancellations was found to be too restrictive–this was before the discovery of the Green-Schwarz anomaly cancellation mechanism. Allowing for Green-Schwarz anomaly cancellation mechanism allows one to construct interesting models in 6 dimensions which are free of local anomalies but could potentially suffer from global anomalies. The modifications of the computation in [6] due to the presence of Green-Schwarz mechanism is relatively straightforward as we will see below.

We start with a theory with no local anomalies, possibly using the Green-Schwarz mechanism. We focus on the gauge groups $SU(2), SU(3)$ and G_2 which could in principle still have $\mathbf{Z_{12}}, \mathbf{Z_6}, \mathbf{Z_3}$ global gauge anomalies. Let α be the generator of the global anomaly group, with $\alpha^n = 1$ for $n = 12, 6, 3$ respectively. Then for each representation R there is an integer $k(R)$ defined mod n where the global transformation leads to $\alpha^k(R)$ change in the phase of the fermion determinant. The condition for absence of global gauge anomalies is that

$$\sum_i k(R_i^+) - \sum_j k(R_j^-) = 0 \qquad \mathrm{mod} \quad n.$$

We are interested in finding $k(R)$.

Let us consider the $SU(3)$ case first. This is essentially a special case considered in [6] as noted above. The only novelty here is that we will use Green-Schwarz mechanism to cancel anomaly and we thus obtain a well defined perturbative $SU(3)$ theory by including a term $B \wedge tr F^2$ in the action where B is an anti-symmetric tensor field which transforms under the gauge transformation. Let us imbed $SU(3)$ in $SU(4)$ where we *include* in the

action the $B \wedge tr F^2$ term needed to cancel local anomaly for $SU(3)$. Then the consideration of [6] go through with the only modification being that the variation of the B field modifies the expression for anomaly to

$$\exp(i \int_{D_7} \gamma_7) \rightarrow \exp(i \int_{D_7} [\gamma_7 - a\gamma_3 tr F^2]$$

for some fixed constant a which makes the integrand vanish on $\partial D_7 = S^6$. Thus the computation reduces once again to an integral on S^7. If we consider the modified 7-form

$$\tilde{\gamma}_7 = \gamma_7 - a\gamma_3 tr F^2$$

it is again proportional to $Tr(g^{-1}dg)^7 + d\eta$ for some η and so the same considerations as in [6] go through.

Let us consider the case of the fundamental representation for $SU(3)$. Using (1.1) and (1.2) and since in the standard embedding of $SU(3)$ in $SU(4)$ we have $\mathbf{4} \rightarrow \mathbf{3} + \mathbf{1}$ we learn that $k_3 = 1$. Moreover for the adjoint of $SU(4)$ we get $A = 8$. Since the adjoint of $SU(4)$ decomposes as $\mathbf{8} + \mathbf{3} + \overline{\mathbf{3}} + 1$ of $SU(3)$ we learn that $k_8 + 2k_3 = 8$ mod 6. We thus have $k_8 = 6 = 0$ mod 3. We can obtain the results for other representations in a similar way (see next section); in particular we find that for the 6 dimensional representation $k_6 = -1$.

We now wish to extend this result to the case of G_2 and $SU(2)$. Since the canonical homomorphism $\pi_6(SU(3)) \rightarrow \pi_6(G_2)$ is onto and using the decompositions $7 \rightarrow \mathbf{3} + \overline{\mathbf{3}} + 1$ and $14 \rightarrow 8 + 3 + \overline{\mathbf{3}}$, we learn that $k_7 = 1$ and $k_{14} = 1$ mod 3 (similarly we can compute k for other representations as will be discussed in the next section). For the case of $SU(2)$ and using the fact that the homomorphism $\pi_6(SU(2)) \rightarrow \pi_6(SU(3))$ is onto we learn that $k_2 = 2$ and $k_3 = 8$ mod 12 (one can use the pseudo-reality to consider the analog of half-hypermultiplets for $SU(2)$ fundamentals in which case we get a square root of the phase for this anomaly).

Now let us specialize the above results to the case of $N = 1$ gauge theories. In this case the gluinos are S^+ spinors in the adjoint representation while the matter fermions is in the S^- representation. Restricting our attention to n_2 doublets of $SU(2)$, n_3 triplets and n_6 sextets of $SU(3)$ and n_7 fundamentals of G_2 we learn that the consistency condition for absence of global gauge anomalies are

$$SU(2): \quad 4 - n_2 = 0 \quad \text{mod } 6$$

$$SU(3): \quad n_3 - n_6 = 0 \quad \text{mod } 6 \tag{1.3}$$

$$G_2: \quad 1 - n_7 = 0 \quad \text{mod } 3$$

It is a simple exercise to check that all the known models of heterotic $E_8 \times E_8$ or $SO(32)$ theory compactified on $K3$ (which yield $N = 1$ in $d = 6$) are consistent with the above restrictions imposed by cancellation of global gauge anomalies.

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2. F-theory Realization of Anomaly

Let us discuss how the F-theory descriptions "knows" about the global anomalies. The fact that geometry can know about such subtle field theory facts has already been observed in [9] where a five dimensional geometry was shown to be "aware" of a Z_2 valued theta angle coming from $\pi_4(SU(2))$. Similarly the geometry was shown to be "aware" of a Z_2 gauge anomaly considered in five dimensional compactifications in [10].

Here we will consider $N = 1$ theories constructed as a local model in F-theory where the geometry consists of an elliptic CY 3-fold with a non-compact 2 dimensional base. Another way of viewing the same theory is as type IIB compactification on $2_{\mathbf{C}}$ -dimensional (complex) base manifold B where the complex coupling τ can make $SL(2,\mathbf{Z})$ jumps. Since the theory is a consistent compactification one would expect it to be free of both local and global anomalies. As we shall see the absence of global anomalies follows from the fact that the two dimensional compact part of the D7-brane in B has *integer valued* self-intersection. Below we will denote by D this two dimensional subspace of the D7-brane. In general the D7-brane may have a multiple number of components in which case we denote them by D_a. Different components of D7-branes may carry different gauge groups G_a. Here we are generalizing the notion of D-brane to include more general groups as allowed by Kodaira classification of singularity in F-theory [2] which would correspond to non-perturbative enhancements of gauge symmetry from the type IIB perspective.

We first start with reviewing the Green-Schwarz mechanism in six dimensions (we will follow in this discussion [11].). To cancel the anomaly in six dimensions via the Green-Schwarz mechanism a certain 8-form

$$I^{(8)} = (\mathrm{tr}R^2)^2 + \frac{1}{6}\mathrm{tr}R^2 \sum_a X_a^{(2)} - \frac{2}{3} \sum_a X_a^{(4)} + 4 \sum_{a<b} Y_{ab} \qquad (2.1)$$

should be factorizable $X_8 = \Omega_{ij} X_i^{(4)} X_j^{(4)}$. The polynomials $X_a^{(n)}$ and Y_{ab} are given as follows

$$X_a^{(n)} = \mathrm{Tr}F_a^n - \sum_R n_{R_a} \mathrm{tr}_{R_a} F_a^n$$

$$Y_{ab} = \sum_{R_a, R_b'} n_{R_a R_b'} \mathrm{tr}_{R_a} F_a^2 \mathrm{tr}_{R_b'} F_b^2 \ .$$

The Tr denotes the trace in the adjoint representation, tr_{R_a} stands for the trace in the representation R_a. The number of matter multiplets in the representation R_a is denoted by n_{R_a} and the number of matter multiplets in the mixed representation – by $n_{R_a, R_a'}$. Ω

is an $n+1 \times n+1$ matrix, where n denotes the number of tensor multiplets. The anomaly is canceled by adding to action a gauge non-invariant term $\int \Omega_{ij} B_i \wedge X_j^{(4)}$.

It is also convenient to introduce the coefficients A_{R_a} and y_{R_a}, that appear in the decomposition $\mathrm{tr}_{R_a} F^4 = A_{R_a} \mathrm{tr} F^4 + y_{R_a} (\mathrm{tr} F^2)^2$ assuming R_a has two independent order four invariants. Let D_a denote the components of the D-brane worldvolume in the base B, as discussed above. Let K denote the canonical divisor of the base (this is a 2-cycle dual to $-c_1(B)$). It was shown in [11] that the existence of the Green-Schwarz counterterm can be traced to the D-brane worldvolume action integrated over D_a. Using this relation it was demonstrated that there is a relation between geometry of D_a and the representation they carry (which are of the form n_{R_a} and $n_{R_a,R_b'}$)

$$\mathrm{index}(Ad_a) - \sum_R \mathrm{index}(R_a) n_{R_a} = 6(K \cdot D_a)$$

$$y_{Ad_a} - \sum_R y_{R_a} n_{R_a} = -3(D_a \cdot D_a)$$

$$A_{Ad_a} - \sum_R A_{R_a} n_{R_a} = 0 \tag{2.2}$$

$$\sum_{R,R'} \mathrm{index}(R_a)\,\mathrm{index}(R_b')\, n_{R_a R_b'} = (D_a \cdot D_b).$$

In the cases where there are no independent 4-th order Casimirs, as is the case for $SU(2), SU(3), G_2, F_4, SO(8), E_6, E_7, E_8$, the cancellation of local anomalies can always be done with Green-Schwarz mechanism for arbitrary representations. However the second equation above, considering the fact that D^2 is an integer may put some integrality restriction. For the case of $F_4, SO(8), E_6, E_7, E_8$ the y_R are all divisible by 3 and so there is no restriction from the above equation. However for $SU(2)$, $SU(3)$ and G_2 we do get a restriction. We find that

$$SU(2) : 16 - \sum 2y_R n_R = -6D^2$$

$$SU(3) : 18 - \sum 2y_R n_R = -6D^2$$

$$G_2 : 10 - \sum y_R n_R = -3D^2$$

Note that for the case of $SU(N)$, $2y_R$ is an integer. For G_2, y_R is integer for all R. These conditions show that there is a mod 6 restriction[3] in the $SU(2)$ and $SU(3)$ cases and a mod 3 condition in G_2 case. In particular we learn that

$$SU(2) : 4 - \sum 2y_R n_R = 0 \qquad mod\ 6$$

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[3] For $SU(2)$ if we allowed half-hypermultiplets, this would be a mod 12 condition.

$$SU(3) : -\sum 2y_R n_R = 0 \qquad mod\ 6$$

$$G_2 : 1 - \sum y_R n_R = 0 \qquad mod\ 3$$

These are exactly the conditions we found in the previous section (1.3) if we can identify $2y_R$ with $k(R)$ for $SU(2)$ and $SU(3)$ and y_R with $k(R)$ for the G_2 case. For the representations considered in the previous section one can readily check that it agrees and the above formula generalizes it to arbitrary representations (which one can also verify using the techniques of the previous section).

3. $N = 1$ critical theories in 6 dimensions

A necessary condition to have non-trivial field theories in 6 dimensions was studied in [3]. In particular for the case of one tensor multiplet a complete classification for the solutions of this condition was given in [4]. The structure of these solutions is roughly as follows: There are a number of exceptional cases corresponding to groups of lower ranks, where they admit finite number of solutions. Then there are in addition 5 infinite series (three based on $SU(N)$, one on $SO(N)$ and one on $SP(N)$). These 5 infinite series form 3-chains (connected by Higgsing to one another): In one chain we have $SU(N)$ with $2N$ fundamentals, in the second chain we have $SP(N)$ with $N + 8$ fundamentals and $SU(N)$ with $N + 8$ fundamentals and 1 anti-symmetric representation and in the third chain we have $SO(N)$ with $N - 8$ vectors and $SU(N)$ with $N - 8$ fundamentals and one symmetric representation.

Now we ask which of these theories are realized in string theory. Taking into account the anomaly we have discovered, the exceptional cases in [4] are indeed all realized in $E_8 \times E_8$ heterotic string compactified on $K3$. The infinite series cannot all be realized in a compact set up (as the rank of the gauge groups for compactifications are bounded). However the first few elements of 4 of the 5 infinite series can be realized in the $K3$ compactification of $E_8 \times E_8$ heterotic strings. The situation is somewhat analogous to type IIA on $K3$ where the rank of the gauge group is bounded in the compact case, whereas if we consider non-compact situations, such as ALE spaces, one can bypass the bound on the rank of the gauge group. It is not surprising, therefore, that also here this extension can be done by considering a local non-compact situation.

The right setup for this construction turns out to be F-theory on elliptic CY 3-folds with a non-compact base. The strong coupling singularity is reached in this setup when the

divisor D of the D7 brane worldvolume vanishes [2]. Vanishing D implies a local invariance under scale transformation (zero size remains zero after rescaling the overall size of the space) which one can thus interpret as IR fixed points. The condition for vanishing of D implies a certain restriction on the divisor D, namely the first Chern class of its normal bundle should be negative, in other words $D^2 < 0$. As it will become clear from the discussion the self-intersection number D^2 is an important characteristic of the critical theory.

Consider first the groups with vanishing fourth order casimir. The spectrum of such theories can be read from (2.2):

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\begin{aligned}
E_7 &: \ n_{\frac{1}{2}56} = D^2 + 8 = 0, 1, ...7 \quad D^2 = -1, -2, ... -8 \\
E_6 &: \ n_{27} = D^2 + 6 = 0, 1, ...5 \quad D^2 = -1, -2, ... -6 \\
F_4 &: \ n_{26} = D^2 + 5 = 0, 1, ...4 \quad D^2 = -1, -2, ... -5 \\
SO(8) &: \ n_{8_{v,s,c}} = D^2 + 4 = 0, 1, ...3 \quad D^2 = -1, -2, ... -4 \\
G_2 &: \ n_7 = 3D^2 + 10 = 1, 4, 7 \quad D^2 = -1, -2, -3 \\
SU(3) &: \ n_3 = 6D^2 + 18 = 0, 6, 12, \ n_6 = 0 \quad D^2 = -1, -2, -3 \\
SU(2) &: \ n_2 = 6D^2 + 16 = 4, 10 \quad D^2 = -1, -2
\end{aligned}
\tag{3.1}
$$

All the gauge groups that appear in (3.1) are subgroups of E_8. The theories with the same D^2 are Higgsable into each other. For example, for $D^2 = -2$ we have the following sequence $E_7 \to E_6 \to F_4 \to SO(8) \to G_2 \to SU(3) \to SU(2)$. If $D^2 = -3$, the same sequence terminates on $SU(3)$ without any matter. For $SU(6)$ there is an exceptional critical theory with 15 fundamentals and half of the tensor multiplet (corresponding to $D^2 = -1$ [4]).

There is also another finite set of examples of $SO(N)$ gauge groups with $N = 7, 8, 9, ...12$ and $N = 13$. The first set of examples corresponds to $D^2 = -1, -2, -3, -4$, while the case $N = 13$ corresponds to two choices of $D^2 = -2, -4$ [5]. The matter spectrum is given by $N_v = N - 4 + D^2$, $N_s = 16(4 + D^2)/d_s$, where d_s is the dimension of spinor representation. N_s counts the total number of spinors in cases with two kinds of spinor representation (like for $SO(12)$).

All these models can be realized as heterotic compactifications on $K3$ or as F-theory compactification with base space Hirzerbruch surface $\mathbf{F}_n$ with 7-branes (where index $n =$

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[4] This example was also realized in [12].

[5] The examples with $D^2 = -4$ fit into an infinite series to be discussed in this section.

$-D^2)[2][12]^6$. The gauge group comes from the singularity in the elliptic fibration at section at infinity S_∞. The structure of elliptic fibration around S_∞ determines the gauge group and the matter spectrum. The vicinity of the singular locus can be modeled by a normal bundle to S_∞ (the normal bundle in question is $\mathcal{O}_{\mathbf{P}^1}(-n)$). The contraction of section S_∞ to a point corresponds to the strong coupling singularity.

Now we formulate the local model of the IR fixed point for F-theory. Consider the base of the 3-fold to be the total space $X^{(-D^2)}$ of the line bundle $\mathcal{O}_{\mathbf{P}^1}(D^2)$ (for example for $D^2 = -2$ the total space X is the cotangent of P^1). We represent the elliptic fibration in the generalized Weierstrass form

$$y^2 + a_1 xy + a_3 y = x^3 + x^2 a_2 + x a_4 + a_6 \tag{3.2}$$

where y and x are sections of some line bundles $\mathcal{L}^3$ and $\mathcal{L}^2$ on X. The restriction of line bundle $\mathcal{L}$ on $\mathbf{P}^1$ is $\mathcal{O}(2 + D^2)$. Coefficients a_i are sections of the bundles $\mathcal{L}^i$. Locally, around the zero section each of the coefficients a_i has the expansion starting with $z^{\sigma_i} a_{i,\sigma_i}$. Taking into account that z is a section of a line bundle $\mathcal{O}(D^2)$ we arrive at the conclusion that each of the coefficients a_{i,σ_i} is a holomorphic section of line bundle $\mathcal{O}(2i + D^2 i - D^2 \sigma)$. As we will see the condition of having holomorphic section of $\mathcal{O}(2i + D^2 i - D^2 \sigma)$ imposes restrictions on possible values of D^2.

The possible types of Kodaira singularities were analyzed in [12] using the Tate's algorithm [13]. The algorithm proceeds roughly as follows: make a change of coordinates (x, y) to put the singularity in the convenient location, blowup the singularity and then repeat. At each stage of this process, after the change of coordinates has been made, the coefficients in the equation will be divisible by certain powers of z. As a result of applying the Tate's algorithm one gets the divisibility properties of a_i encoded in values of σ_i.

Let us first start with the $SU(2)$ case. In order to get an I_2 singularity in the elliptic fibration the coefficients σ_i should be equal to $(\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_6) = (0, 0, 1, 1, 2)$ or $(\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_6) = (0, 1, 1, 1, 2)$. Both choices lead to the same six dimensional field theories. The condition of having holomorphic section of $\mathcal{O}(2i - ni + \sigma_i n)$ implies that

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[6] The case of $SO(13)$ was not considered in [12]. It can be realised in $E_8 \times E_8$ heterotic compactification by considering $SO(13) \times SO(3) \subset SO(16) \subset E_8$ and choosing the $SO(3)$ gauge bundle with instanton number $(6 + n)$. The spectrum of $SO(13)$ theory is $(2n + 9)13 + \frac{(2n+4)}{4} 64$ and $2n = D^2 = -4, -2$.

$2i + D^2 i - D^2 \sigma > 0$ leading to two choices of $D^2 = -1, -2$. Let us define the local model for $SU(2)$ IR fixed point as

$$y^2 + a_1 xy + za_{3,1}y = x^3 + x^2 a_2 + xza_{4,1} + z^2 a_{6,2} \; . \tag{3.3}$$

This construction can be easily generalized to other models as soon as we know the spectrum of σ_i (see Table 2 in [12]). For example, for $SU(3)$ gauge group $(\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_6) = (0, 1, 1, 2, 3)$ and $(\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_6) = (1, 1, 2, 2, 3)$ for G_2.

As already mentioned, apart from the finite set of exceptional examples that can be realized in heterotic compactification on $K3$ there are 5 infinite series of IR fixed points that cannot be all realized in a compact situation. Also as mentioned above they come in 3 chains. The first few elements of these chains are already realized in the compactification of heterotic string corresponding to the $D^2 = -1, -2, -4$: The $SU(N)$ series with $2N$ fundamental correspond to $D^2 = -2$, the $SP(N)$ with $N+8$ fundamentals and the $SU(N)$ series with $N+8$ fundamentals and one antisymmetric tensor correspond to $D^2 = -1$ and the $SO(N)$ with $N-8$ fundamentals appears at $D^2 = -4$. The series with $SU(N)$ with $N-8$ fundamentals and the symmetric tensor is not realized perturbatively (but if it were it should have appeared at $D^2 = -4$ because it is higgsable to the SO series and Higgsing does not affect the value of D^2).

We now ask if we can realize these infinite series. We consider the local model in F-theory with the non-compact base being the total space of the line bundle $\mathcal{O}(D^2)$ on $\mathbf{P}^1$ considered above. Let us first start with the $SU(N)$ case. The spectrum of σ's can be determined from [12]. Namely, $(\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_6) = (0, 1, k, k, 2k)$ for $N = 2k$ and $(\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_6) = (0, 1, k-1, k, 2k-1)$ for $N = 2k - 1$. For example, for even N the elliptic fibration is given as follows

$$y^2 + a_1 xy + z^k a_{3,k} y = x^3 + x^2 za_{2,1} + xz^k a_{4,k} + z^{2k} a_{6,2k} \; , \tag{3.4}$$

and similar expression in case of odd N. The coefficients a_{i,σ_i} are holomorphic sections of $\mathcal{O}(2i + D^2 i - D^2 \sigma_i)$. The existence of these holomorphic sections implies that $D^2 = -1, -2$. It is clear from the construction that the vanishing cycle is a smooth sphere. Therefore, for $D^2 = -2$ we get a realization of $SU(N)$ IR fixed points with $2N$ fundamentals. Similarly, for $D^2 = -1$ we obtain a realization of $SU(N)$ IR fixed point with $(N+8)$ fundamentals and one antisymmetric tensor.

The $Sp(N)$ ($D^2 = -1$) series of critical theories are realized in a similar manner. The spectra of σ_i are given in Table 2 of [12]. For the case of $SO(N)$, $N > 8$, one finds that as long as $0 > D^2 \geq -4$ using Tate's algorithm one can construct an infinite series of examples. For the case of $D^2 = -4$ this gives the $SO(N)$ series with $N - 8$ fundamentals. For the other cases when $8 \leq N \leq 12$ and $D^2 = -1, -2, -3$ we get the exceptional SO series discussed above. For $N > 13$ and $D^2 = -1, -2, -3$ the singularity of the local model (which would have given spinor in the lower rank SO cases) prevents an interpretation in terms of matter representations but leads to a perfectly sensible critical theory with no field theory realization. Note that in this local model, we can always change the complex structure, which is the analog of "higgsing" and reduce to the phase where there is no strange singularity (basically by decreasing the value of N). Moreover it can be shown that these singularities in complex structure appear at finite distance in Calabi-Yau moduli space [14]. This situation is similar to the case of small E_8 instantons. Note that a small E_8 instanton is the generalization of the matter in the 56 of E_7 to that of E_8 (in the sense that the 56 of E_7 arises from the singularity enhancement to E_8 [15], whereas small E_8 instantons arise from singularity enhancement of E_8 to E_9 [2]). The singularity we encounter here for larger values of N in $SO(N)$ is the generalization of the singularity which leads to spinor matter for $SO(N)$ for $N \leq 13$ (which comes from enhancement of SO singularity to the exceptional series of singularities).

As discussed before we have already constructed all the local SU series allowed in this local setup. The only series we have not constructed using F-theory, which is expected based on field theory analysis of [4], is that of $SU(N)$ with $(N - 8)$ fundamentals and one symmetric tensor. For this case using the results of [11] one sees that this should be possible and that the local model is *not* a normal bundle over $\mathbf{P}^1$, but rather a bundle over $g = 1$ surface with one double point. It should be possible to explicitly construct this series in this local setup.

4. D-brane realization of $N = 1$ six dimensional SCFTs

Two of the series we have discussed seem to have a perturbative D-brane realization. One is the $SU(N)$ series with $N_f = 2N$ and the other is $SO(N)$ with $N - 8$ fundamentals.

We first start with the $SU(N)$ gauge group. Consider total space $X^{(2)}$ of the bundle $\mathcal{O}_{\mathbf{P}^1}(-2)$. This space is the cotangent bundle on $\mathbf{P}^1$ and it has a trivial first Chern class. To realize an $SU(N)$ singularity we wrap N 7-branes over the $D = \mathbf{P}^1$ (zero section). By

doing this we immediately introduce some curvature that has to be canceled by additional 7-branes. Let us denote the additional 7-branes that intersect D as Σ_i $((D \cdot \Sigma_i) = 1)$. Each 7-brane Σ_i intersecting D gives rise to a matter multiplet in the fundamental representation [16]. The canonical class of X with additional 7-branes is equal to

$$12K = ND + \sum_{i=1}^{N_f} \Sigma_i + ... , \tag{4.1}$$

where the dots denote the extra 7-branes that do not intersect with D. The condition that the total canonical class is zero implies that

$$12K \cdot D = 0 = (ND + \sum_{i=1}^{N_f} \Sigma_i) \cdot D = -2N + N_f$$

which leads to $N_f = 2N$.

The next case we wish to consider is for $D^2 = -4$. Consider in particular the case with the $SO(8)$ gauge group without matter. This case can be viewed as T^*P^1/Z_2 orientifold of type IIB [17] [18]. Putting 8 D7-branes cancels the charge due to the orientifold and prevents the coupling from running. Now if we wish to add $N-8$ more D7-branes wrapped on the orientifold $\mathbf{P}^1$, thus getting gauge group $SO(N)$ we must make sure that the extra curvature is cancelled by $N-8$ intersecting D7-branes, just as discussed above for the $SU(N)$ case. This gives rise to $SO(N)$ gauge theory with $N-8$ fundamentals.

5. Field Theory versus Geometry

In this paper we have seen that field theory consideration is rather powerful in predicting and classifying possible SCFT's that arise from string theory compactifications, in that necessary conditions from field theory appear to be sufficient. This was also found to be the case for certain 5-dimensions SCFT's considered in [10].

However if we are interested in studying questions beyond just mere classification of SCFT's and in particular for the properties of conformal theories themselves and the possible branches of such theories, geometry has the upper hand. Not only geometry will show whether the necessary conditions for the existence of SCFT's from the QFT considerations are sufficient, but it will also point to the existence of fixed points which have no field theory interpretation such as the three series we have discovered (see also the example in [9][19]). Another aspect involves slight deformations away from SCFT's.

For example for the SCFT's in 5 and 6 dimensions there are tensionless strings, whose properties are best understood in the context of geometry. This is for example manifest in the constructions of BPS states for such theories [20][21]. Another arena where geometry has the upper hand is in the questions of transitions from one branch to another where the physical question of transition from one branch to another is mapped to a concrete geometric question. We strongly believe that the geometric description is the most powerful way to think about SCFT's and their properties and that there are many more physical properties in store for us which we need to decode from the geometric realization of SCFT's.

6. Acknowledgments

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